Math 524 Exam 7 Solutions

1. Given any 2×2 matrix A, we consider the usual three systems, as below. If possible, produce five such matrices A, subject to the restrictions given. (I) $x(n) = Ax(n-1)$, (II) $dx/dt = Ax$, (III) $d^2x/dt^2 = Ax$.

   (a) (I) and (II) stable or neutral, (III) unstable

   (b) (I) and (III) stable or neutral, (II) unstable

   (c) (II) and (III) stable or netural, (I) unstable

   (d) all three systems unstable

   (e) all three systems stable

   (I) is unstable precisely when some $|\lambda| > 1$; (II) is unstable precisely when some $Re\lambda > 0$; (III) is unstable precisely when some λ is not real and nonpositive. Many solutions are possible, of course. (A): $\begin{pmatrix} -0.1-0.1i & 0 \\ 0 & -0.1-0.1i \end{pmatrix}$; (B): impossible, since (III) stable or neutral implies (II) is as well; (C): $\begin{pmatrix} -2 & 0 \\ 0 & -3 \end{pmatrix}$; (D): $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$; (E): $\begin{pmatrix} -0.1 & 0 \\ 0 & -0.2 \end{pmatrix}$

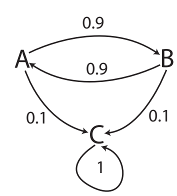
2. If possible, produce five Markov chains, subject to the following conditions:

   (a) irreducible, aperiodic, and recurrent

   This is the nice, ergodic, case. Any regular Markov chain will work, for instance.

   (b) reducible, with at last one state periodic and at least one state transient

   The following example has A, B both periodic and transient:



   (c) aperiodic and recurrent, but reducible

   One simple solution has $A \to A$ and $B \to B$, each with probability 1.

   (d) irreducible and recurrent, but at least one state periodic

   One simple solution has $A \to B$ and $B \to A$, each with probability 1.

   (e) irreducible and aperiodic, but at least one state transient

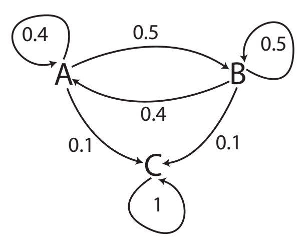
   This is the hardest question on this exam. If the Markov chain has finitely many states, then it is not possible. Proof: the time-average steady state must have probability 0 for every transient state, since otherwise it would be revisited infinitely often. Since the system is irreducible, the

transient state has at least one other state that feeds into it. The time-average steady state must have probability 0 for THIS state as well, since otherwise it would feed into the transient state infinitely often. But then we repeat, and eventually get that every state has probability 0 in the time-average steady state, an impossibility.
However, it is possible to have this phenomenon. For example, we consider infinitely many states A_i, for all integer i. $A_i \to A_{i+1}$ with probability 0.9, and $A_i \leftarrow A_{i+1}$ with probability 0.1. This moves toward increasing subscripts with high probability. In fact, EVERY state is transient, since there is a positive probability it will not be revisited ever again.

3. Consider the difference equation $x(n) = 5x(n-1) - 6x(n-2)$, with initial conditions $x(0) = 6, x(1) = 17$. Convert this into a 2×2 first-order problem, then solve it to get the general solution $x(n)$.

   Setting $y(n) = x(n-1)$, we have $\begin{pmatrix} y(n) \\ x(n) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -6 & 5 \end{pmatrix} \begin{pmatrix} y(n-1) \\ x(n-1) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -6 & 5 \end{pmatrix}^{n-1} \begin{pmatrix} y(1) \\ x(1) \end{pmatrix}$. This is diagonalizable: $\begin{pmatrix} 0 & 1 \\ -6 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix}$. Hence $\begin{pmatrix} y(n) \\ x(n) \end{pmatrix} =$ $\begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}^{n-1} \begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} y(1) \\ x(1) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 2^{n-1} & 0 \\ 0 & 3^{n-1} \end{pmatrix} \begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 6 \\ 17 \end{pmatrix} =$ $= \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 2^{n-1} & 0 \\ 0 & 3^{n-1} \end{pmatrix} \begin{pmatrix} 1 \\ 5 \end{pmatrix} = \begin{pmatrix} 2^{n-1}+5\cdot 3^{n-1} \\ 2^n+5\cdot 3^n \end{pmatrix}$. Hence $x(n) = 2^n + 5 \cdot 3^n$.

4. Consider the Markov chain pictured below. If the initial distribution is starting in A, i.e. $(1, 0, 0)^T$, find (approximately) the distribution after 12 time steps. You may use the approximation that $(0.9)^{12} \approx 2/7$.



   This has transition matrix $M = \begin{pmatrix} 0.4 & 0.4 & 0 \\ 0.5 & 0.5 & 0 \\ 0.1 & 0.1 & 1 \end{pmatrix}$. This has known eigenvalue 1, since the column sums are 1. The trace is 1.9, the determinant is 0, so the remaining two eigenvalues have sum 0.9 and product 0; hence the three eigenvalues are $0, 0.9, 1$, with eigenvectors $(-1, 1, 0), (4, 5, -9), (0, 0, 1)$ respectively. Hence, the general solution is $\alpha(1)^n \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \beta(0.9)^n \begin{pmatrix} 4 \\ 5 \\ -9 \end{pmatrix} +$ $\gamma(0)^n \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$. The initial condition corresponds to $\alpha = 1; \beta = 1/9; \gamma = -5/9$. Evaluating at $n = 12$ gives the approximate solution $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + (1/9)(2/7) \begin{pmatrix} 4 \\ 5 \\ -9 \end{pmatrix} =$ $\begin{pmatrix} 8/63 \\ 10/63 \\ 45/63 \end{pmatrix}$